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Brans–Dicke gravity theory from topological gravity

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ABSTRACT

We consider a model that suggests a mechanism by which the four dimensional Brans–Dicke gravity theory may emerge from the topological gravity action.

To achieve this goal, both the Lie algebra and the symmetric invariant tensor that define the topological gravity Lagrangian are constructed by means of the Lie algebra S-expansion procedure with an appropriate abelian semigroup S.

1. Introduction

It has been known for a long time that in General Relativity the spacetime is a dynamical object which has independent degrees of freedom, and is governed by dynamical equations, namely the Einstein field equations. This means that in General Relativity the geometry is dynamically determined. Therefore, the construction of a gauge theory of gravity requires an action that does not consider a fixed space-time background. An action for gravity fulfilling these conditions was proposed long ago by Chamseddine [1,2].

A.H. Chamseddine [1,2] constructed actions for topological gravity in all dimensions. In these references it was shown that the odd-dimensional theories are based on Chern–Simons forms with the gauge groups taken to be $ISO(2n,1)$ or $SO(2n+1,1)$ or $SO(2n,2)$ depending on the sign of the cosmological constant. The use of the Chern–Simons form was essential so as to have a gauge invariant action without constraints.

The even-dimensional theories use, in addition to the gauge fields, a scalar multiplet in the fundamental representation of the gauge group. For even-dimensional spaces there is no natural geometric candidate such as the Chern–Simons form. The wedge product of n of the field strengths can made the required $2n$-form in a $2n$-dimensional space-time. To form a group invariant $2n$-form, the n-product of the field strength is not enough, but will require in addition the introduction of a scalar field ϕ^a in the fundamental representation.

If topological gravity theories are to provide the appropriate gauge-theory framework for the gravitational interaction, then these theories must satisfy the correspondence principle, namely they must be related to General Relativity or to Brans–Dicke theory.

It is the purpose of this paper to show that Brans–Dicke theory emerges as the $\ell \to 0$ limit of a topological gravity theory invariant under the so called AdS-Lorentz algebra ($AdS\,\mathcal{L}_4$) [3,4] (see also [5]). Here ℓ is a length scale—a coupling constant that characterizes different regimes within the theory. The AdS-Lorentz algebra, on the other hand, is constructed from the AdS algebra and a particular semigroup S by means of the S-expansion procedure introduced in Refs. [6,9]. The field content induced by AdS-Lorentz includes the vielbein e^a, the spin connection ω^{ab} and the extra bosonic fields k^{ab}, ϕ^{ab}, h^{ab}, ϕ^a.

This paper is organized as follows: In Section 2 we briefly review some aspects of (i) topological gravity, (ii) the S-expansion procedure and (iii) the Brans–Dicke gravity theory. An explicit action for four-dimensional gravity is considered in Section 3 where the Lie algebra S-expansion procedure is used to obtain an $AdS\,\mathcal{L}_4$-invariant topological gravity action. The weak coupling constant limit of this action is then shown to yield the Brans–Dicke action. The work concludes with a comment and with Appendices A and B, where the case of a topological gravity invariant under the $\mathfrak{B}_5$ algebra [10] is considered.

2. Topological gravity, S-expansion method and Brans–Dicke gravity theory

In this section we shall review some aspects of (i) topological gravity, (ii) the S-expansion procedure and (iii) the Brans–Dicke gravity theory.

2.1. Topological gravity

In Refs. [1,2] Chamseddine constructed actions for topological gravity in all dimensions. For odd dimensions, the action is given by

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$$S_{2n+1} = k \int_{M_{2n+1}} L_{2n+1} \tag{1}$$

where

$$L_{2n+1} = \sum_{l=0}^{n} \alpha_l \varepsilon_{a_1 a_2 \cdots a_{2n+1}} R^{a_1 a_2} \cdots R^{a_{2l-1} a_{2l}} e^{a_{2l+1}} \cdots e^{a_{2n+1}} \tag{2}$$

is a Chern–Simons form, and $R^{ab} = d\omega^{ab} + \omega^{ac}\omega_c{}^b$ with $a = 0, 1, 2 \cdots 2n$.

The even-dimensional theories use, in addition to the gauge fields, a scalar multiplet in the fundamental representation of the gauge groups. The $2n$-dimensional action is given by [1,2]

$$S_{TG}^{(2n)} = k \int_{M_{2n}} \varepsilon_{a_1 a_2 \cdots a_{2n+1}} \phi^{a_{2n+1}} R^{a_1 a_2} \cdots R^{a_{2n-1} a_{2n}}, \tag{3}$$

where $R^{ab} = d\omega^{ab} + \omega^{ac}\omega_c{}^b$ with $a = 0, 1, 2 \cdots 2n$, and $M_{2n} = \partial M_{2n+1}$. This action was obtained from a Chern–Simons action using a dimensional reduction method (for detail see Appendix B).

2.2. S-expansion procedure

In this subsection we shall review the main aspects of the S-expansion procedure and their properties introduced in Ref. [6] (see also [7,8]).

Let $S = \{\lambda_\alpha\}$ be an abelian semigroup with 2-selector $K_{\alpha\beta}{}^\gamma$ defined by

$$K_{\alpha\beta}{}^{\gamma} = \begin{cases} 1 & \text{when } \lambda_\alpha \lambda_\beta = \lambda_\gamma \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

and $\mathfrak{g}$ a Lie (super)algebra with basis $\{\mathbf{T}_A\}$ and structure constant $C_{AB}{}^C$,

$$[\mathbf{T}_A, \mathbf{T}_B] = C_{AB}{}^C \mathbf{T}_C. \tag{5}$$

Then it may be shown that the product $\mathfrak{G} = S \times \mathfrak{g}$ is also a Lie algebra with structure constants $C_{(A,\alpha)(B,\beta)}{}^{(C,\gamma)} = K_{\alpha\beta}{}^\gamma C_{AB}{}^C$,

$$[\mathbf{T}_{(A,\alpha)}, \mathbf{T}_{(B,\beta)}] = C_{(A,\alpha)(B,\beta)}{}^{(C,\gamma)} \mathbf{T}_{(C,\gamma)}. \tag{6}$$

The proof is direct and may be found in Ref. [6].

Definition 1. Let S be an abelian semigroup and $\mathfrak{g}$ a Lie algebra. The Lie algebra $\mathfrak{G}$ defined by $\mathfrak{G} = S \times \mathfrak{g}$ is called S-expanded algebra of $\mathfrak{g}$.

When the semigroup has a zero element $0_S \in S$, it plays a somewhat peculiar role in the S-expanded algebra. The above considerations motivate the following definition:

Definition 2. Let S be an abelian semigroup with a zero element $0_S \in S$, and let $\mathfrak{G} = S \times \mathfrak{g}$ be an S-expanded algebra. The algebra obtained by imposing the condition $0_S \mathbf{T}_A = 0$ on $\mathfrak{G}$ (or a subalgebra of it) is called 0_S-reduced algebra of $\mathfrak{G}$ (or of the subalgebra).

An S-expanded algebra has a fairly simple structure. Interestingly, there are at least two ways of extracting smaller algebras from $S \times \mathfrak{g}$. The first one gives rise to a *resonant subalgebra*, while the second produces reduced algebras. In particular, a resonant subalgebra can be obtained as follows:

Let $g = \bigoplus_{p \in I} V_p$ be a decomposition of g in subspaces V_p, where I is a set of indices. For each $p, q \in I$ it is always possible to define $i_{(p,q)} \subset I$ such that

$$[V_p, V_q] \subset \bigoplus_{r \in i_{(p,q)}} V_r. \tag{7}$$

Now, let $S = \bigcup_{p \in I} S_p$ be a subset decomposition of the abelian semigroup S such that

$$S_p \cdot S_q \subset \bigcup_{r \in i_{(p,q)}} S_p. \tag{8}$$

When such subset decomposition $S = \bigcup_{p \in I} S_p$ exists, then we say that this decomposition is in resonance with the subspace decomposition of g, $g = \bigoplus_{p \in I} V_p$.

The resonant subset decomposition is crucial in order to systematically extract subalgebras from the S-expanded algebra $G = S \times g$, as is proven in the following theorem

Theorem IV.2. *(See Ref. [6].) Let $g = \bigoplus_{p \in I} V_p$ be a subspace decomposition of g, with a structure described by Eq. (7), and let $S = \bigcup_{p \in I} S_p$ be a resonant subset decomposition of the abelian semigroup S, with the structure given in Eq. (8). Define the subspaces of $\mathfrak{G} = S \times \mathfrak{g}$,*

$$W_p = S_p \times V_p, \quad p \in I. \tag{9}$$

Then,

$$\mathfrak{G}_R = \bigoplus_{p \in I} W_p \tag{10}$$

is a subalgebra of $G = S \times g$.

Proof. The proof may be found in Ref. [6]. □

Definition 3. The algebra $G_R = \bigoplus_{p \in I} W_p$ is called a Resonant Subalgebra of the S-expanded algebra $G = S \times g$.

A useful property of the S-expansion procedure is that it provides us with an invariant tensor for the S-expanded algebra $\mathfrak{G} = S \times \mathfrak{g}$ in terms of an invariant tensor for $\mathfrak{g}$. As shown in Ref. [6] Theorem VII.2 provides a general expression for an invariant tensor for a 0_S-reduced algebra.

Theorem VII.2. *(See Ref. [6].) Let S be an abelian semigroup with nonzero elements λ_i, $i = 0, \cdots, N$ and $\lambda_{N+1} = 0_S$. Let $\mathfrak{g}$ be a Lie algebra of basis $\{\mathbf{T}_A\}$, and let $\langle \mathbf{T}_{A_n} \cdots \mathbf{T}_{A_n} \rangle$ be an invariant tensor for $\mathfrak{g}$. The expression*

$$\langle \mathbf{T}_{(A_1,i_1)} \cdots \mathbf{T}_{(A_n,i_n)} \rangle = \alpha_j K_{i_a \cdots i_n}{}^j \langle \mathbf{T}_{A_1} \cdots \mathbf{T}_{A_n} \rangle \tag{11}$$

where α_j are arbitrary constants, corresponds to an invariant tensor for the 0_S-reduced algebra obtained from $\mathfrak{G} = S \times \mathfrak{g}$.

Proof. The proof may be found in Section 4.5 of Ref. [6]. □

2.3. Brans–Dicke gravity theory

The starting point for Brans and Dicke is the idea of Mach, that the phenomenon of inertia ought to arise from accelerations with respect to the general mass distribution of the universe. Thus the inertial masses of the various elementary particles ought not to be fundamental constants, but should rather represent the particles interaction with some cosmic field. But the absolute scale of the elementary particle masses can be measured only the measuring gravitational accelerations Gm/r^2, so an equivalent conclusion is that the gravitational constant G ought to be related to the average value of a scalar field ϕ, which is coupled to the mass density of the universe [11].

The simplest generally covariant field equation for such a scalar field would be

$$\Box^2 \phi = 4\pi T^{\mu}_{M\mu} \tag{12}$$

where $\Box^2\phi = \phi^{//\rho}_{//\rho}$ is the invariant D'Alembertian, λ is a coupling constant, and $T^{\mu\nu}_M$ is the energy–momentum tensor of the matter of the universe. We can make a rough estimate of the average value of ϕ by computing the central potential of a gas sphere with the cosmic mass density $\rho \sim 10^{-29}$ g/cm^3 and radius equal to the apparent radius of the universe $R \sim 10^{28}$ cm. This gives an average value [11]

$$\langle\phi\rangle \sim \lambda\rho R^2 \sim \lambda 10^{27}\ \mathrm{g\,cm^{-1}} \tag{13}$$

Note that $10^{27}\ \mathrm{g\,cm^{-1}}$ is reasonably close to the constant $1/G = 1.35 \times 10^{28}\ \mathrm{g\,cm^{-1}}$; hence we normalize ϕ so that

$$\langle\phi\rangle \simeq \frac{1}{G} \tag{14}$$

and (13) then shows that λ is a dimensionless number of order unity. These considerations led Brans and Dicke to suggest that the correct field equations for gravitation are obtained by replacing G with $1/\phi$ and including an energy–momentum tensor $T^{\mu\nu}_\phi$ for the ϕ-field in the source of the gravitational field.

In the first order formalism, the Brans–Dicke action is given by [12],

$$S_{BD} = \int_{M_4} \varepsilon_{abcd}\phi R^{ab}e^c e^d \tag{15}$$

Setting, for notational simplicity $\phi = e^{-2\varphi}$ we have

$$S_{BD} = \int_{M_4} \varepsilon_{abcd}e^{-2\varphi} R^{ab}e^c e^d \tag{16}$$

Let us vary e^d and φ, we find respectively

$$2\varepsilon_{abcd}e^{-2\varphi}R^{ab}e^c = 0 \tag{17}$$

$$-2\varepsilon_{abcd}e^{-2\varphi}R^{ab}e^c e^d = 0 \tag{18}$$

The variation with respect to ω^{ab} leads to

$$\varepsilon_{abcd}T^c e^d = \varepsilon_{abcd}d\phi e^c e^d \tag{19}$$

where the new feature is the appearance of a nonzero torsion. From (19) it follows that

$$T^a = d\varphi e^a \tag{20}$$

Hence the spin connection ω^{ab} is non-Riemannian.

Let us set

$$\omega^{ab} = \mathring{\omega}^{ab} + \gamma^{ab} \tag{21}$$

where $\mathring{\omega}^{ab}$ is the Riemannian part of ω^{ab}, that is it satisfies $T^a(\mathring{\omega}^{ab}) = 0$. In this case (20) becomes

$$\gamma_c^{ab} = \delta_c^a\partial^b\varphi - \delta_c^b\partial^a\varphi \tag{22}$$

or as a 1-form

$$\gamma^{ab} = 2e^{[a}\partial^{b]}\varphi \tag{23}$$

Substituting into (17) and (18) we find [12]

$$\mathring{R}^a_b - \frac{1}{2}\delta^a_b\mathring{R} + \Omega^a_b - \frac{1}{2}\delta^a_b\Omega = 0 \tag{24}$$

where $\mathring{R}^a_b = R^a_b(\mathring{\omega})$ is the curvature 2-form in terms of the Riemannian connection $\mathring{\omega}$ and where

$$\Omega^a_b = -\mathring{D}_b\partial^a\varphi - \frac{1}{2}\delta^a_b\mathring{D}_c\partial^c\varphi + \delta^a_b\partial^c\varphi\partial_c\varphi - \partial^a\varphi\partial_b\varphi$$
$$\Omega = -3\left(\mathring{D}_m\partial^m\varphi - \partial^m\varphi\partial_m\varphi\right) \tag{25}$$

Substituting (25) in (24) we find the Einstein equation for the Brans–Dicke theory in the second order formalism:

$$\mathring{R}^a_b - \frac{1}{2}\delta^a_b\mathring{R} = \mathring{D}_b\partial^a\varphi - \delta^a_b\mathring{D}_m\partial^m\varphi + 3\left(\partial^a\varphi\partial_b\varphi - \frac{1}{2}\delta^a_b\partial^m\varphi\partial_m\varphi\right) \tag{26}$$

If one converts (26) to the world-tensor formalism, one finds the alternative form appearing in standard literature [12].

3. Brans–Dicke action from topological gravity

In this section we show how to recover the Brans–Dicke gravity action from topological gravity.

The action for four-dimensional topological gravity can be written as [1,2,13] (see also Appendix B)

$$S^{(4)}_{TG} = \int_{M_4=\partial M_5} \langle\phi F F\rangle \tag{27}$$

where ϕ is a scalar field in the fundamental representation of the gauge group $SO(4,2)$ or $SO(5,1)$ (anti-de Sitter or de Sitter) and $F = dA + AA$. Connection with gravity is made through the identification

$$A^{ab} = \omega^{ab}; \qquad A^{a5} = e^a, \quad a = 0,1,2,3,4 \tag{28}$$

The Lagrangian (27) is invariant under the AdS algebra. This algebra is crucial, since it permits the interpretation of the gauge fields e^a and ω^{ab} as the five dimensional vielbein and the five dimensional spin connection in five dimensions, respectively. It is, however, not the only possible choice. As was explicitly shown in Ref. [10], there exist other Lie algebras that also allow for a similar identification.

Following the definitions of Ref. [6] let us consider the S-expansion of the Lie AdS algebra using as a semigroup $S^{(2)}_{\mathcal{M}} = \{\lambda_0, \lambda_1, \lambda_2\}$ endowed with the multiplication rule $\lambda_\alpha\lambda_\beta = \lambda_{\alpha+\beta}$ if $\alpha + \beta \le 2$; $\lambda_\alpha\lambda_\beta = \lambda_{\alpha+\beta-2}$ if $\alpha + \beta > 2$. After extracting a resonant subalgebra, one finds the $AdS\,\mathcal{L}_4$-algebra [14], which coincides with the $\mathfrak{so}(D-1,1) \oplus \mathfrak{so}(D-1,2)$ algebra of Ref. [3,4], whose D-dimensional generators satisfy the following commutation relations

$$\begin{aligned}
&[J_{ab}, J_{cd}] = \eta_{bc}J_{ad} + \eta_{ad}J_{bc} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}\\
&[J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b; \qquad [P_a, P_b] = Z_{ab}\\
&[J_{ab}, Z_{cd}] = \eta_{bc}Z_{ad} + \eta_{ad}Z_{bc} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac}\\
&[Z_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b\\
&[Z_{ab}, Z_{cd}] = \eta_{bc}Z_{ad} + \eta_{ad}Z_{bc} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac}
\end{aligned} \tag{29}$$

where $a, b, c, d = 0, 1, \cdots, D-1$.

This algebra was also re-obtained in Ref. [15] from Maxwell algebra through a procedure known as deformation of algebras.

Using Theorem VII.2 of Ref. [6], it is possible to show that the only non-vanishing components of an invariant tensor for the 5-dimensional $AdS\,\mathcal{L}_4$ algebra $(so(4,2) \oplus so(4,1))$ are given by

$$\begin{aligned}
&\langle J_{ab}J_{cd}P_e\rangle = \frac{4}{3}\alpha_1 l^3\varepsilon_{abcde}\\
&\langle Z_{ab}Z_{cd}P_e\rangle = \frac{4}{3}\alpha_1 l^3\varepsilon_{abcde}\\
&\langle J_{ab}Z_{cd}P_e\rangle = \frac{4}{3}\alpha_1 l^3\varepsilon_{abcde}
\end{aligned} \tag{30}$$

where α_1 is an arbitrary constant of dimensions $[\text{length}]^{-3}$, and $a, b, c, d, e = 0, 1, 2, 3, 4$. It is interesting to note that the 4-dimensional $AdS\,\mathcal{L}_4$ algebra is given by $so(3,2) \oplus so(3,1)$.

In order to write down a topological gravity Lagrangian for the five dimensional $AdS\,\mathcal{L}_4$ algebra, we start from the $AdS\,\mathcal{L}_4$-valued one-form gauge connection

$$A = \frac{1}{l} e^a P_a + \frac{1}{2}\omega^{ab} J_{ab} + \frac{1}{2} k^{ab} Z_{ab}, \quad a, b = 0, 1, 2, 3, 4. \tag{31}$$

The associated two-form curvature is then given by

$$F = \frac{1}{2} R^{ab} J_{ab} + \frac{1}{l}\left(T^a + k^a{}_b e^b\right) P_a + \frac{1}{2}\left(D_\omega k^{ab} + k^a{}_c k^{cb} + \frac{1}{l^2} e^a e^b\right) Z_{ab}, \tag{32}$$

and the 0-form scalar field is

$$\phi = \frac{1}{2}\phi^{ef} J_{ef} + \frac{1}{l}\phi^e P_e + \frac{1}{2} h^{ef} Z_{ef} \tag{33}$$

Consistency with the dual procedure of S-expansion in terms of the Maurer–Cartan forms [9] demands that ϕ^e inherits units of length from the vielbein; that is why it is necessary to introduce the l parameter again, this time associated with ϕ^e.

It is interesting to observe that J_{ab} are still Lorentz generators, but P_a are no longer AdS boosts; in fact, $[P_a, P_b] = Z_{ab}$. However, e^a still transforms as a vector under Lorentz transformations, as it must be in order to recover gravity in this scheme.

Using the dual procedure of S-expansion in terms of the Maurer–Cartan forms [9], we find that the topological gravity action invariant under the $AdS\,\mathcal{L}_4$ algebra is given by

$$\begin{aligned} S^{(AdS\,\mathcal{L}_4)}_{BD-(4)} = \int_{M_4} \frac{\alpha_1 l^2}{3}\varepsilon_{abcde}\Big\{\phi^e\Big[&R^{ab}R^{cd} \\ &+ 2R^{ab}\left(D_\omega k^{cd} + k^c{}_f k^{fd} + \frac{1}{l^2}e^c e^d\right) \\ &+ \left(D_\omega k^{ab} + k^a{}_f k^{fb} + \frac{1}{l^2}e^a e^b\right) \\ &\times \left(D_\omega k^{cd} + k^c{}_f k^{fd} + \frac{1}{l^2}e^c e^d\right)\Big] \\ &+ 2h^{ab}\Big[\left(D_\omega k^{cd} + k^c{}_f k^{fd} + \frac{1}{l^2}e^c e^d\right)\left(T^e + k^e{}_f e^f\right) \\ &+ R^{cd}\left(T^e + k^e{}_f e^f\right)\Big] \\ &+ 2\phi^{ab}\Big[\left(D_\omega k^{cd} + k^c{}_f k^{fd} + \frac{1}{l^2}e^c e^d\right)\left(T^e + k^e{}_f e^f\right) \\ &+ R^{cd}\left(T^e + k^e{}_f e^f\right)\Big]\Big\} \end{aligned} \tag{34}$$

which can be rewritten in the following way

$$\begin{aligned} S^{(AdS\,\mathcal{L}_4)}_{BD-(4)} = \int_{M_4} &\frac{\alpha_1 l^2}{3}\varepsilon_{abcde}\phi^e R^{ab}R^{cd} + \frac{2\alpha_1 l^2}{3}\varepsilon_{abcde}\phi^e R^{ab} D_\omega k^{cd} \\ &+ \frac{2\alpha_1}{3}\varepsilon_{abcde}\Big\{\phi^e R^{ab} e^c e^d + \frac{1}{2l^2}\phi^e e^a e^b e^c e^d \\ &+ \phi^e\left(D_\omega k^{ab}\right)e^c e^d + \phi^e\left(k^2\right)^{ab} e^c e^d\Big\} \\ &+ \frac{2\alpha_1 l^2}{3}\varepsilon_{abcde}\Big\{\phi^e R^{ab}\left(k^2\right)^{cd} + \phi^e\left(D_\omega k^{ab}\right)\left(k^2\right)^{cd} \\ &+ \frac{1}{2}\phi^e\left(k^2\right)^{ab}\left(k^2\right)^{cd} + \frac{1}{2}\phi^e\left(D_\omega k^{ab}\right)\left(D_\omega k^{cd}\right)\Big\} \\ &+ \frac{\alpha_1 l^2}{3}\varepsilon_{abcde}\Big\{2h^{ab}\Big[\left(D_\omega k^{cd} + \left(k^2\right)^{cd} + \frac{1}{l^2}e^c e^d\right) \\ &\times\left(T^e + k^e{}_f e^f\right) + R^{cd}\left(T^e + k^e{}_f e^f\right)\Big] \\ &+ 2\phi^{ab}\Big[\left(D_\omega k^{cd} + \left(k^2\right)^{cd} + \frac{1}{l^2}e^c e^d\right)\left(T^e + k^e{}_f e^f\right) \\ &+ R^{cd}\left(T^e + k^e{}_f e^f\right)\Big]\Big\} \end{aligned} \tag{35}$$

where $(k^2)^{ab} = k^a{}_f k^{fb}$.

The action (35) was obtained from (27) via the S-expansion procedure. Therefore e^a and ω^{ab} in (35) are one-forms pulled baked in 4-dimensions. For simplicity we consider Eq. (35) for the case $k^{ab} = 0$ and $T^a = 0$:

$$S^{(4)}_{AdS\,L_4} = \int_{M_4 = \partial M_5} \alpha_1 \varepsilon_{abcde}\left(\frac{l^2}{3}R^{ab}R^{cd}\phi^e + \frac{2}{3}R^{ab}e^c e^d \phi^e + \frac{1}{3l^2}e^a e^b e^c e^d \phi^e\right) \tag{36}$$

where $R^{ab} = d\omega^{ab} + \omega^a{}_c\omega^{cb}$ and $a, b, c, d, e : 0, 1, 2, 3, 4$. Using the decomposition $e^a = (e^i, e^4)$ with $i = 0, 1, 2, 3$, together with decomposition (54) (see Appendix B), we find

$$S^{(4)}_{AdS\,L_4} = \int_{M_4 = \partial M_5} \alpha_1 \phi \varepsilon_{ijkl}\left(\frac{l^2}{3}\bar{R}^{ij}\bar{R}^{kl} + \frac{2}{3}\bar{R}^{ij}e^k e^l + \frac{1}{3l^2}e^i e^j e^k e^l\right) \tag{37}$$

where $\bar{R}^{ij} = R^{ij} + \frac{1}{l^2}e^i e^j$. Here ω^{ij}_μ and $e_\mu{}^i$, with $\mu = 0, 1, 2, 3$ and $i = 0, 1, 2, 3$, can be interpreted as the 4-dimensional spin connection and the vierbein respectively.

From (37) we can see that for small values of l^2 we find

$$S^{(4)}_{AdS\,L_4} = \frac{2}{3}\alpha_1 \int_{M_4 = \partial M_5} \varepsilon_{ijkl}\left(\phi R^{ij}e^k e^l + \frac{3}{2l^2}\phi e^i e^j e^k e^l\right) \tag{38}$$

which correspond to the Brans–Dicke action with a cosmological term. On the other hand, when ϕ be a constant, the action (37) corresponds to the Einstein–Hilbert action with a cosmological term.

Concluding remarks

From (35) we can see that if ϕ^e were a constant, then (i) the first and the second terms would be an exact form; (ii) the third piece of $L^{(AdS\,\mathcal{L}_4)}_{BI-(4)}$ corresponds to, except for numerical factors, the Lagrangian (29) of Ref. [16] (see also [17]).

We have shown in this work that the S-expansion procedure makes possible to recover the four-dimensional Brans–Dicke gravity theory with a cosmological term from a topological gravity theory invariant under the so called AdS-Lorentz algebra.

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Appendix A. Topological gravity and $\mathfrak{B}_5$ algebra

Following the definitions of Ref. [6], let us consider the S-expansion of the Lie algebra $SO(4,2)$ using as semigroup $S_E^{(3)}$. After extracting a resonant subalgebra and performing its 0_S-reduction, one finds a new Lie algebra, call it $\mathfrak{B}_5$, with the desired properties. In simpler terms, consider the Lie algebra generated by $\{J_{ab}, P_a, Z_{ab}, Z_a\}$, where these new generators can be written as $J_{ab} = \lambda_0 \otimes \tilde{J}_{ab}$, $Z_{ab} = \lambda_2 \otimes \tilde{J}_{ab}$, $P_a = \lambda_1 \otimes \tilde{P}_a$, $Z_a = \lambda_3 \otimes \tilde{P}_a$.

Here $\tilde{J}_{ab}$ and $\tilde{P}_a$ correspond to the original generators of $SO(4,2)$, and the λ_α belong to a discrete, abelian semigroup. The semigroup elements $\{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ are *not* real numbers and they are *dimensionless* (see [10]). In this particular case, they obey the multiplication law given by

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha+\beta \leq 4, \\ \lambda_4, & \text{when } \alpha+\beta > 4. \end{cases} \tag{39}$$

Using Theorem VII.2 of Ref. [6], it is possible to show that the only non-vanishing components of an invariant tensor for the $\mathfrak{B}$ algebra are given by [6]

$$\begin{aligned} \langle J_{a_1a_2} J_{a_3a_4} P_{a_5} \rangle &= \alpha_1 \frac{4l^3}{3} \epsilon_{a_1\cdots a_5}, \\ \langle J_{a_1a_2} J_{a_3a_4} Z_{a_5} \rangle &= \alpha_3 \frac{4l^3}{3} \epsilon_{a_1\cdots a_5}, \\ \langle J_{a_1a_2} Z_{a_3a_4} P_{a_5} \rangle &= \alpha_3 \frac{4l^3}{3} \epsilon_{a_1\cdots a_5}, \end{aligned} \tag{40}$$

where α_1 and α_3 are arbitrary independent constants of dimensions $[\text{length}]^{-3}$.

In order to write down a Lagrangian for the $\mathfrak{B}$ algebra, we start from the one-form gauge connection

$$A = \frac{1}{2}\omega^{ab} J_{ab} + \frac{1}{l} e^a P_a + \frac{1}{2} k^{ab} Z_{ab} + \frac{1}{l} h^a Z_a, \tag{41}$$

the two-form curvature

$$\begin{aligned} F &= \frac{1}{2} R^{ab} J_{ab} + \frac{1}{l} T^a P_a + \frac{1}{2}\left(D_\omega k^{ab} + \frac{1}{l^2} e^a e^b \right) Z_{ab} \\ &\quad + \frac{1}{l}\left(D_\omega h^a + k^a{}_b e^b \right) Z_a. \end{aligned} \tag{42}$$

and the 0-form scalar field

$$\phi = \frac{1}{2}\phi^{ef} J_{ef} + \frac{1}{l}\phi^e P_e + \frac{1}{2}\bar{\phi}^{ef} Z_{ef} + \frac{1}{l}\bar{\phi}^e Z_e \tag{43}$$

Consistency with the dual procedure of S-expansion in terms of the Maurer–Cartan forms [9] demands that h^a, ϕ^e and $\bar{\phi}^e$ inherit units of length from the vielbein; that is why it is necessary to introduce the l parameter again, this time associated with h^a, ϕ^e and $\bar{\phi}^e$.

It is interesting to observe that $\boldsymbol{J}_{ab}$ are still Lorentz generators, but P_a are no longer AdS boosts; in fact, $[P_a, P_b] = Z_{ab}$. However, e^a still transforms as a vector under Lorentz transformations, as it must be in order to recover gravity in this scheme.

A direct calculation shows that it is possible to write down a Lagrangian for topological gravity in four dimensions for the $\mathfrak{B}_5$ algebra as

$$\begin{aligned} L^{(4)}_{\text{TG}} &= \frac{\alpha_1 l^2}{3} \varepsilon_{abcde} \left[2\phi^{ab} R^{cd} T^e + R^{ab} R^{cd} \phi^e \right] \\ &\quad + \frac{\alpha_3 l^2}{2} \varepsilon_{abcde} \Big[\phi^{ab} R^{cd} \left(D_\omega h^e + k^e{}_g e^g \right) + \frac{1}{2} R^{ab} R^{cd} \bar{\phi}^e \\ &\quad + \phi^{ab} D_\omega k^{cd} T^e + \phi^e R^{ab} D_\omega k^{cd} + \bar{\phi}^{ab} R^{cd} T^e \Big] \\ &\quad + \alpha_3 \varepsilon_{abcde} \left[\phi^e R^{ab} e^c e^d + \phi^{ab} e^c e^d T^e \right] \end{aligned} \tag{44}$$

Here it is necessary to notice an important point: The Lagrangian is split into two independent pieces, one proportional to α_1 and the other to α_3.

Following the same procedure used in the last section we find that the piece proportional to α_3 contains the Brans–Dicke term $\varepsilon_{ijkl}\phi R^{ij} e^k e^l$ plus non-linear couplings between the curvature and the bosonic "matter" fields k^{ij}, ϕ^{ij}, $\bar{\phi}^{ij}$, $\bar{\phi}^i$ and h^i, where the parameter l can be interpreted as a kind of coupling constant.

Appendix B. Topological gravity as a gauged Wess–Zumino–Witten term

Chern–Simons gravity theories have been extended by using transgression forms instead of Chern–Simons forms as actions [18–25]. Chern–Simons and transgression gravity theories are valid only in odd-dimensions and in order to have a well defined even-dimensional theory it would be necessary some kind of dimensional reduction or compactification.

In Ref. [26], subsequently Ref. [27–29] and most recently Ref. [30] it was pointed out that Chern–Simons theories are connected with some even-dimensional structures known as gauged Wess–Zumino–Witten ($gWZW$) terms.

The connection between this even-dimensional structure and the Chern–Simons gravity theories suggest that this mechanism could be regarded as an alternative to compactification or dimensional reduction.

From Ref. [31] we can see that

$$\begin{aligned} Q_{2n+1}\left(A^Z, A\right) &= Q_{2n+1}\left(A^Z, F^Z\right) - Q_{2n+1}(A, F) \\ &\quad - dB_{2n}\left(A^Z, A\right) \end{aligned} \tag{45}$$

where the connections A^Z and A are related by a gauge transformation given by $A^Z = z^{-1}(d+A)z$ with $z = e^{-\phi^a P_a}$ and $A^Z = \frac{1}{2} W^{ab} J_{ab} + V^a P_a = V + W$.

The term $Q_{2n+1}(A, F)$ corresponding to the Lagrangian for $(2n+1)$-dimensional Chern–Simons gravity for the one-form connection A, is given by (see for example Appendix of Ref. [13])

$$\begin{aligned} Q_{2n+1}(A, F) &= \varepsilon_{a_1a_2\cdots a_{2n+1}} R^{a_1a_2} \cdots R^{a_{2n-1}a_{2n}} e^{a_{2n+1}} \\ &\quad - n(n+1) d \left\{ \int_0^1 dt t^n \left\langle R_t{}^{n-1} \omega e \right\rangle \right\} \end{aligned} \tag{46}$$

where $R_t = d\omega + t\omega^2$.

If A^z and A are given by $A = e^a P_a + \frac{1}{2}\omega^{ab} J_{ab} = e + \omega$ and $A^Z = V^a P_a + \frac{1}{2} W^{ab} J_{ab} = V + W$, where $V^a = e^a + D_\omega \phi^a$ and $W^{ab} = \omega^{ab}$, then $Q_{2n+1}(A^Z, F^Z)$ is given by (see for example Ref. [13])

$$\begin{aligned} Q_{2n+1}\left(A^Z, F^Z\right) &= \varepsilon_{a_1a_2\cdots a_{2n+1}} R^{a_1a_2} \cdots R^{a_{2n-1}a_{2n}} e^{a_{2n+1}} \\ &\quad + \varepsilon_{a_1a_2\cdots a_{2n+1}} R^{a_1a_2} \cdots R^{a_{2n-1}a_{2n}} D\phi^{a_{2n+1}} \\ &\quad - n(n+1) d \left\{ \int_0^1 dt t^n \left\langle R_t{}^{n-1} \omega e \right\rangle \right\} \\ &\quad - n(n+1) d \left\{ \int_0^1 dt t^n \left\langle R_t{}^{n-1} \omega D\phi \right\rangle \right\} \end{aligned} \tag{47}$$

On the other hand $B_{2n}(A, A^Z)$ is given by [13]

$$B_{2n}\left(A^Z, A\right) = -n(n+1) \int_0^1 dt t^n \left\langle R_t{}^{n-1} \omega D\phi \right\rangle \tag{48}$$

Introducing (46), (47) and (48) into (45) we find

$$Q_{2n+1}\left(A^Z, A\right) = \varepsilon_{a_1....a_{2n+1}} R^{a_1a_2}...R^{a_{2n-1}a_{2n}} D\phi^{a_{2n+1}} \tag{49}$$

using the Bianchi identity $DR^{ab} = 0$ we can write

$$Q_{2n+1}\left(A^Z, A\right) = d\left[\varepsilon_{a_1....a_{2n+1}} R^{a_1a_2}...R^{a_{2n-1}a_{2n}} \phi^{a_{2n+1}}\right] \tag{50}$$

which proves that the action for topological gravity in $2n$-dimensions, found in Refs. [1,2], is a *gWZW* term given by

$$\begin{aligned} S^{(2n)}_{gWZW}\left(A^Z, A\right) &= k \int\limits_{M_{2n+1}} Q_{2n+1}\left(A^Z, A\right) \\ &= k \int\limits_{M_{2n}=\partial M_{2n+1}} \varepsilon_{a_1....a_{2n+1}} R^{a_1a_2}...R^{a_{2n-1}a_{2n}} \phi^{a_{2n+1}}. \end{aligned} \tag{51}$$

This means that the action for topological gravity in $2n$-dimensions, introduced in Ref. [1], is a gauged Wess–Zumino–Witten term. This means that the $2n$-dimensional topological gravity is described by the dynamics of the boundary of a $(2n+1)$ Chern–Simons gravity.

In the particular case $n = 2$ we have

$$\begin{aligned} S^{(4)}_{gWZW}\left(A^Z, A\right) &= k \int\limits_{M_5} Q_{2n+1}\left(A^Z, A\right) \\ &= k \int\limits_{M_4=\partial M_5} \varepsilon_{abcde} R^{ab} R^{cd} \phi^e \end{aligned} \tag{52}$$

where R^{ab} is a 2-form pulled backed in four-dimensions. This means that Eq. (52) can be written as

$$S^{(4)} = k \int\limits_{M_4=\partial M_5} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{abcde} R^{ab}_{\mu\nu} R^{cd}_{\rho\sigma} \phi^e \tag{53}$$

where $\mu, \nu, \rho, \sigma : 0, 1, 2, 3$ and $a, b, c, d, e : 0, 1, 2, 3, 4$.

Using the decomposition

$$\begin{aligned} &\omega^{ab}_\mu = \left(\omega^{ij}_\mu, \omega^{i4}_\mu = \lambda e_\mu{}^i\right), \quad i, j = 0, 1, 2, 3 \\ &\phi^a = \left(\phi^i, \phi^4 \equiv \phi\right) \end{aligned} \tag{54}$$

and rotating the basis in such a way that in each point of space the field ϕ^a has components $\phi^4 \equiv \phi$, $\phi^i = 0$, we have

$$\bar{R}^{ij} = R^{ij} + \lambda^2 e^i e^j \tag{55}$$

with $R^{ij} = d\omega^{ij} + \omega^i{}_k \omega^{kj}$. This means that the action (52) can be written as

$$S^{(4)} = k \int\limits_{M_4} \phi \varepsilon_{ijkl} \bar{R}^{ij} \bar{R}^{kl}. \tag{56}$$

Now ω^{ij}_μ with $\mu = 0, 1, 2, 3$ and $i = 0, 1, 2, 3$, can be interpreted as the four dimensional spin connection and $e_\mu{}^i$ can be interpreted as the vierbein. This means that the decomposition (54) is consistent with the remaining $so(3, 2)$ or $so(4, 1)$ invariance.

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